Take-home exam in “Ergodic theory and Fractal geometry” (80763),

Spring 2015

Instructions

- From the moment you read the exam you have one week to complete it.
- In order to receive a grade you must submit the exam by October 1, 2015. Submit a hard-copy to me in person or by email (mhochman@math.huji.ac.il).
- You may read any text you want before or during the exam, including presentations of the material covered in the course, but if you come across a solution to one of the problems in the exam you may NOT read it!
- There are 4 problems. You may either solve 3 out of the 4 problems, in which case each is worth $33\frac{1}{3}$ points; or you may solve all 4, then each is worth 30 points, and the final grade will be $\max\{\text{exam grade}, 100\}$. The passing grade for BA students is 60, for MA students it is 80.
- It is recommended to review the material *before* starting the exam!

Do not hesitate to contact me with questions.

GOOD LUCK.

Problem 1

Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function and μ a compactly supported, invariant and ergodic probability measure for f. Suppose that $f' \neq 0$ on the support of μ and that $\lambda = \int \log |f'| d\mu > 0$. Show that

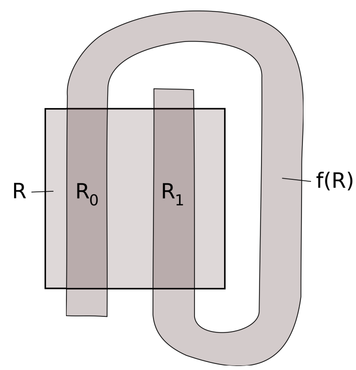
$$\dim \mu = \frac{h_\mu(f)}{\lambda}$$

i.e. the pointwise dimension of μ exists μ-a.e. and is equal to $h_\mu(f)/\lambda$.

Note: f is conformal, so as a map $\mathbb{R}^2 \to \mathbb{R}^2$ its derivative is a similarity (a scalar times a 2×2 rotation matrix).

Problem 2

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism that maps the unit square $R = [0,1]^2$ to the region depicted in the image below, so that $R \cap f(R)$ consists of two rectangles R_0, R_1 of width α and height 1. Assume that on R_i the function f^{-1} has the form $x \mapsto \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix} x + c_i$, with $0 < \alpha < 1 < \beta$ and $c_0, c_1 \in \mathbb{R}^2$.



1. Describe the intersections $R \cap f(R) \cap f^2(R)$ and, in general, $\bigcap_{i=0}^n f^i(R)$. (A drawing and short explanation suffice). Similarly describe $R \cap f^{-1}(R)$ and $\bigcap_{i=-n}^0 f^i(R)$, and finally, describe $\bigcap_{i=-n}^n f(R)$.

2. Let
$$\Omega = \bigcap_{i=-\infty}^{\infty} f^i(R)$$
This is a compact f-invariant set. Show that (Ω, f) is isomorphic to $(\{0,1\}^{\mathbb{Z}}, S)$ where S is the shift, i.e. construct a homeomorphism $\pi : \{0,1\}^{\mathbb{Z}} \to \Omega$ such that $\pi S = Sf$.

3. Apply the Oseledets theorem to $\Omega \ni x \mapsto Df(x) \in GL(2, \mathbb{R})$ and describe the resulting exponents and subspaces.

4. Show that if $\mu \in \mathcal{P}(\Omega)$ is an f-invariant and ergodic probability then it has pointwise dimension a.e., and express this dimension in terms of $h_\mu(f)$ and α, β. (Hint: the analysis is similar to

what we did for automorphisms of the 2-torus; if you take this approach you can formulate an appropriate Brin-Katok lemma without proving it. Alternatively you can use the model (Ω, S) directly).

Problem 3

Let $A \in SL_d(\mathbb{Z})$ be hyperbolic (no eigenvalues of modulus 1) and $T = T_A$ the induced map on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Let W^+, W^-, be the sum of the generalized eigenspaces of A associated to eigenvalues of magnitude > 1 (respectively < 1). Let $\mathcal{W}^\pm$ be the corresponding partitions of $\mathbb{T}^d$ (i.e. $y \in \mathcal{W}^\pm(x)$ if $y = x + w \bmod 1$ for some $w \in \mathcal{W}^\pm$). Let μ be an invariant and ergodic probability measure for T.

Show that the following are equivalent:

1. $h_\mu(T) = 0$
2. $\dim(\mu, x) = 0$ μ-a.e.
3. For every partition $\mathcal{V}$ subordinate to $\mathcal{W}^+$ (equivalently $\mathcal{W}^-$),

$$\mu_x^{\mathcal{V}} = \delta_x \qquad \mu\text{-a.e. } x$$

Problem 4

Let $U \subseteq \mathbb{R}^d$ be open and $f : U \to U$ a diffeomorphism (so f^{-1} is also differentiable). Let μ be a compactly supported ergodic probability measure for f. Apply Oseledets's theorem to $x \mapsto Df(x) \in GL_d(\mathbb{R})$ to obtain $\lambda_1 < \ldots < \lambda_k$ and linear subspaces $\{0\} \neq V_1^x \leq V_2^x \leq \ldots \leq V_k^x = \mathbb{R}^d$ such that for μ-a.e. x we have

$$\frac{1}{n} \log \|Df^n(x)v\| \to \lambda_i \qquad \text{for all } v \in V_x^i \setminus V_x^{i-1}$$

and $Df(x)V_i^x = V_i^{f(x)}$ (in the equation we set $V_0^x = \{0\}$. Note that the notation differs frmo that int he proof of the theorem, it is more in line with the statement of Corollary 2.3 in Sarig's notes).

Suppose that $g : U \to U$ is a diffeomorphism also preserving μ. Apply the Oseledets theorem to $x \mapsto Dg(x)$ to obtain corresponding $\sigma_1 < \ldots < \sigma_m$ and $W_1^x \leq \ldots \leq W_m^x$.

Assume now that f, g commute, that is $f \circ g = g \circ f$.

1. Show that $V_i^{g(x)} = Dg(x)V_i^x$ (μ-a.e. x).
2. Assume now that each V_i and W_i is 1-dimensional, so $m = k = d$. Show that there is a permutation π of $\{1, \ldots, k\}$ such that $V_i^x = W_{\pi(i)}^x$.
3. Under the assumptions of (2), must we have $\pi = \mathit{identity}$? Must we have $\lambda_i = \sigma_{\pi(i)}$?